Test 2A, Math 1410

Name: Key

PID Number: ____________________

fix quest 2, part (a) to say "that is, find $\frac{dV}{dT}$"

I pledge that I have neither given nor received any unauthorized assistance on this exam.

____
(*signature*)

DIRECTIONS

1. Show all of your work. A correct answer with insufficient work will lose points, as will an answer with incorrect notation.
2. Clearly indicate your answer by putting a box around it.
3. Calculators are allowed on this exam.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge and write your ID on both pages.
6. Number of questions = 10. Total Points = 100.

PID Number: Key

Test A

1. (8 points) Calculate y' if

$$y = \tan\left[\ln\left(\frac{1}{x^3}\right)\right]$$

$$y = \tan\left(\ln x^{-3}\right)$$

$$y = \tan\left(-3 \ln x\right)$$

$$\boxed{y' = \sec^2(-3 \ln x) \cdot \left(\frac{-3}{x}\right)}$$

2. (10 points) The law of Graham-Squire states that the following relationship holds between P (Pressure), V (Volume), and T (Temperature) of air inside a sphere. (Assume that P, V, and T are all positive numbers.)

$$\frac{PV^3}{T^2} = K$$

Where K is some constant.

(8)

(a) Assuming that the pressure is fixed (i.e. P is held constant), find the rate of change of the volume with respect to the temperature.

$$\frac{d}{dT}\left(V^3 = \frac{K}{P} \cdot T^2 \right)$$

$$3V^2 \cdot \frac{dV}{dT} = \frac{K}{P} \cdot (2T)$$

$$\boxed{\frac{dV}{dT} = \frac{2KT}{3V^2 P}}$$

✓✓ (b) Assuming that P is held constant, use your answer from (a) to explain why an increase in temperature will result in an increase in the volume. Hint: It may help to think about the volume function.

$\frac{dV}{dT}$ is positive, so the Volume function is increasing.

Thus an increase in temperature will result in an increase in Volume.

3. (14 points) For the equation $x^2 + xy + \sin^2 y = 1$, find $\frac{dy}{dx}$ and the equation for the tangent line at the point (1,0).

$$\frac{d}{dx}\left(x^2 + xy + (\sin y)^2 = 1 \right)$$

$$\Leftrightarrow \quad 2x + (y + xy') + 2 \sin y \cos y \, y' = 0$$

$$xy' + 2 \sin y \cos y \, y' = -2x - y$$

$$y'(x + 2 \sin y \cos y) = -2x - y$$

$$\boxed{y' = \frac{-2x - y}{x + 2 \sin y \cos y}}$$

$$y - y_1 = m(x - x_1)$$

$$x_1 = 1$$

$$y_1 = 0$$

$$y - 0 = -2(x - 1)$$

$$y'\Big|_{(1,0)} = \frac{-2(1) - 0}{1 + 2 \sin 0 \cos 0}$$

$$\boxed{y = -2x + 2}$$

$$\Rightarrow m = -2$$

4. (12 points) Use calculus to find the x-coordinate(s) where $y = \dfrac{x^2-3}{x-2}$ has a horizontal tangent line.

$$y' = \frac{(x-2)(2x) - (x^2-3)(1)}{(x-2)}$$

(8) 6 $$y' = \frac{2x^2-4x-x^2+3}{(x-2)}$$

$$y' = \frac{x^2-4x+3}{x-2}$$

(2) Set $y'=0$, then $0 = \dfrac{x^2-4x+3}{x-2}$

$$\Longleftrightarrow x^2-4x+3=0$$

(4) $$(x-3)(x-1)=0$$

So at $\boxed{x=3 \text{ and } x=1}$ y will have a horizontal tangent.

5. (8 points) Find $\frac{d}{dx}(\arccos(x)\sqrt{1-x^2})$. Simplify your answer.

$$= \frac{-1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} + \arccos x \cdot \frac{1}{\cancel{2}}(1-x^2)^{-1/2} \cdot (-\cancel{2}x)$$

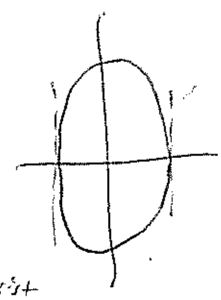
$$\boxed{= -1 - \frac{x(\arccos x)}{\sqrt{1-x^2}}}$$

6. (4 points) State if the following is true or false. Give a brief explanation to justify your answer.

"The graph of $9x^2 + 4y^2 = 36$ is continuous and has no sharp corners or cusps, therefore the derivative $\frac{dy}{dx}$ exists for all points on the graph."

False. $9x^2+4y^2=36$ is an ellipse,

so at the points where $y=0$ (that is, $(-2,0)$ and $(2,0)$) the graph will have a vertical tangent line, so $\frac{dy}{dx}$ will not exist at those points.



Also: $\frac{d}{dx}(9x^2+4y^2=36) \Longrightarrow 18x+8yy'=0 \Rightarrow y' = \frac{-9x}{4y}$

due at $y=0$

7. (10 points) Using only the derivatives for $\sin x$ and $\cos x$, as well as either the chain rule or the quotient rule, prove that

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{(\cos x)^2}$$

$$= \frac{\sin x}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \cdot \sec x$$

8. (10 points) Use differentials to approximate the change in the area of a circle as the radius changes from 5 cm to 5.3 cm.

$dy = f'(x_1)\,dx$

$f(x) = \pi x^2$

$x_1 = 5$

$dx = 0.3\text{ cm}$

$f'(x) = 2\pi x\ \text{cm}^2/\text{cm}$

$dy = 2\pi(5)\,\text{cm}^2/\text{cm} \cdot 0.3\text{ cm}$

$\boxed{dy = 3\pi\ \text{cm}^2}$

9. (12 points) Let $A(x) = \dfrac{f(g(x))}{h(x)}$. Find $A'(0)$ if the following values hold for f, g, and h:

$$\begin{array}{ccc} f(0) = 4 & g(0) = 1 & h(0) = -2 \\ f(1) = -1 & g(1) = 6 & h(1) = 7 \\ f'(0) = -7 & g'(0) = 3 & h'(0) = 2 \\ f'(1) = -3 & g'(1) = -5 & h'(1) = 0 \end{array}$$

Simplify your answer. Note: You may not need all of the values above.

$$A'(x) = \frac{f'(g(x)) \cdot g'(x) \cdot h(x) - f(g(x)) \cdot h'(x)}{(h(x))^2}$$

$$\Rightarrow A'(0) = \frac{f'(g(0)) \cdot g'(0) \cdot h(0) - f(g(0)) \cdot h'(0)}{(h(0))^2}$$

$$= \frac{f'(1) \cdot 3 \cdot (-2) - f(1) \cdot 2}{(-2)^2}$$

$$= \frac{-3 \cdot (-6) - (-1)2}{4}$$

$$= \frac{18 + 2}{4}$$

$$= \boxed{5}$$

10. (12 points) Use logarithmic differentiation to calculate $f'(x)$ for

$$f(x) = (\cot x)^{1/x}$$

Simplify your answer.

$$y = (\cot x)^{1/x}$$

$$\Rightarrow \ln y = \ln \cot x^{1/x}$$

$$\ln y = \left(\frac{1}{x}\right) \cdot \ln(\cot x)$$

$$\frac{d}{dx}\left(\ln y = \frac{1}{x} \cdot \ln(\cot x)\right)$$

$$\frac{1}{y} y' = \frac{-1}{x^2} \cdot \ln(\cot x) + \frac{1}{x} \cdot \frac{1}{\cot x} \cdot -\csc^2 x$$

$$y' = (\cot x)^{1/x}\left[\frac{-\ln(\cot x)}{x^2} + \frac{-1}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\sin^2 x}\right]$$

$$y' = -(\cot x)^{1/x}\left[\frac{\ln(\cot x)}{x^2} + \frac{1}{x \sin x \cos x}\right]$$

Extra Credit (2 points): Is the following equation True or False? If true briefly explain why. If false, either explain why or give a counterexample.

$$\frac{d}{dx}[f(\sqrt{x})g(x)] = [f'(x) \cdot \frac{1}{2}x^{(-1/2)}]g(x) + g'(x)f(x)$$

False. These should be $\sqrt{x}$